

A PARRONDO PARADOX IN RELIABILITY THEORY*

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Abstract

Parrondo's paradox arises in sequences of games in which a winning expectation may be obtained by playing the games in a random order, even though each game in the sequence may be lost when played individually. We present a suitable version of Parrondo's paradox in reliability theory involving two systems in series, the units of the first system being less reliable than those of the second. If the first system is modified so that the distributions of its new units are mixtures of the previous distributions with equal probabilities, then under suitable conditions the new system is shown to be more reliable than the second in the "usual stochastic order" sense.

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1 Introduction

In their simpler formulation, Parrondo's games are composed of two 'atomic' games: playing each atomic game individually the player is led to lose, whereas he may expect to win if the games are played randomly. These games were first described by Parrondo [12] at a workshop in 1996, and later studied by Harmer and Abbott [8], [9], and Harmer *et al.* [11]. The counterintuitive behavior of Parrondo's games has been exhibited under several sets of rules, depending on the player's current capital or the history of the game. Parrondo's games can also be thought of as the discrete-time version of flashing Brownian ratchets, namely the periodical or random combination of Brownian motion in an asymmetric potential and of Brownian motion in a flat potential (see Parrondo *et al.* [13], Astumian [4], [5], Allison and Abbott [1], for instance). As with Brownian ratchets, Parrondo's games are gaining increasing attention in the literature: a feasible device for reproducing Parrondo's games has been proposed by Allison and Abbott [2] and the role of chaos in Parrondo's games has been investigated by Arena *et al.* [3]. We also refer the reader to the comprehensive review by Harmer and Abbott [10], and the list of articles shown in Web page [16].

The result implied by Parrondo's games, by which several losing strategies can be turned into a winning strategy by careful combination, is often referred to as "Parrondo's paradox". As one might expect, this surprising and intriguing fact has also kindled the interest of the popular press (see Blakeslee [6]). However, to avoid illusions it should be pointed out that Parrondo's phenomenon is not applicable to several real-life games (such as those played in casinos), or in the stock market. Indeed, as shown for instance by Philips and Feldman [14], Parrondo's paradox

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does not lead to any contradiction. The right combination of two losing strategies can be a winning strategy, without violating any mathematical or physical property. As in many other paradoxes in probability, the crucial intuition (or its lack) centers on the independence (or the lack of it) of the games. Parrondo's games are composed of *non-independent* losing games (which are very far from games played in the real world), and it should not be surprising that a suitable combination of non-independent losing games may produce a winning game. Nevertheless, even if Parrondo's paradox does not lead to contradictions, it does help to formulate and solve many interesting problems, such as finding improved or optimal strategies in sequences of games.

The aim of this note is to present a suitable version of Parrondo's paradox in the field of reliability theory. We compare pairs of systems in series with two components, the units of the first system being less reliable than those of the second. We study the effect of a modification in the first system consisting in the random choice of its components: Assuming that each unit is chosen randomly from a set of components identical to the previous ones, then the distributions of the new units are mixtures of the previous distributions. With such randomization we obtain a system that, under suitable conditions, is shown to be more reliable than the second one, although the single starting units are less reliable than those of the second system. This result, proved in Section 2, should not be considered contradictory, and indeed in the reliability theory literature it is well known that mixtures of distributions do not preserve certain properties. For instance, whereas mixtures of decreasing failure rate distributions are always decreasing, very often mixtures of increasing failure rate distributions may also decrease (see Finkelstein and Esaulova [7], for instance).

Finally, in Section 3 we give a sketch of a Parrondo's game based on the outcomes of lifetimes of systems in series.

2 The paradox

Consider two systems, \mathcal{S}_X and \mathcal{S}_Y , both formed by two units connected in series. Let X_1 and X_2 be non-negative random variables that describe the lifetimes of the units of system \mathcal{S}_X . We denote by $\overline{F}_i(t) = \Pr\{X_i > t\}$, $t \geq 0$, the absolutely continuous survival function of X_i , $i = 1, 2$. The random lifetime of \mathcal{S}_X is thus given by $X := \min\{X_1, X_2\}$, and possesses the survival function

$$\overline{F}(t) := \overline{F}_1(t) \overline{F}_2(t), \quad t \geq 0. \quad (1)$$

Similarly, let Y_1 and Y_2 be non-negative random variables that describe the lifetimes of the series units of system \mathcal{S}_Y . They are characterized by absolutely continuous survival functions $\overline{G}_1(t)$ and $\overline{G}_2(t)$, $t \geq 0$. Hence, the lifetime of \mathcal{S}_Y is $Y := \min\{Y_1, Y_2\}$, and has the survival function

$$\overline{G}(t) := \overline{G}_1(t) \overline{G}_2(t), \quad t \geq 0. \quad (2)$$

The random variables X_1, X_2, Y_1, Y_2 are assumed to be independent, all with support $(0, +\infty)$, i.e. with $\overline{F}_i(t) > 0$ and $\overline{G}_i(t) > 0$ for all $t \geq 0$, $i = 1, 2$.

Furthermore, assume that the components of system \mathcal{S}_Y are more reliable than those of \mathcal{S}_X with respect to the "usual stochastic order" (for a detailed study of such stochastic order see Shaked and Shanthikumar [15]): we have $X_i \leq_{\text{st}} Y_i$ for $i = 1, 2$, i.e. $\overline{F}_i(t) \leq \overline{G}_i(t)$ for all $t \geq 0$, $i = 1, 2$. In other words, for all times $t \geq 0$ lifetime Y_i is more likely than X_i to have a duration larger than t . From (1) and (2), we see that system \mathcal{S}_Y is more reliable than \mathcal{S}_X , since $\overline{F}(t) \leq \overline{G}(t)$ for all $t \geq 0$, and then $X \leq_{\text{st}} Y$.

Let us now compare the above systems when the selection of the units of \mathcal{S}_X is randomized. Assume that both components of the first system can be chosen randomly from two collections

of units characterized by i.i.d. random lifetimes distributed as X_1 and X_2 , respectively. In other words, we consider a new series system, \mathcal{S}_{X^*} say, whose i -th component has survival function $\overline{F}_i^*(t)$ given by a mixture with equal probabilities of the survival functions of X_1 and X_2 , i.e.

$$\overline{F}_i^*(t) := \frac{\overline{F}_1(t) + \overline{F}_2(t)}{2}, \quad t \geq 0, \quad i = 1, 2.$$

Consequently, the new system lifetime X^* has survival function

$$\overline{F}^*(t) := \overline{F}_1^*(t) \overline{F}_2^*(t) = \left[\frac{\overline{F}_1(t) + \overline{F}_2(t)}{2} \right]^2, \quad t \geq 0. \quad (3)$$

From (1) and (3) we have

$$\overline{F}^*(t) - \overline{F}(t) = \left[\frac{\overline{F}_1(t) - \overline{F}_2(t)}{2} \right]^2 \geq 0, \quad t \geq 0,$$

so that $X^* \geq_{\text{st}} X$. System \mathcal{S}_{X^*} is thus more reliable than \mathcal{S}_X . In other terms, the random choice of the components increases the system lifetime in the usual stochastic order sense.

Bearing in mind that lifetimes X_1 and X_2 are stochastically smaller than Y_1 and Y_2 , respectively, we now face the following problem: to establish that, under certain conditions, system \mathcal{S}_{X^*} is more reliable than \mathcal{S}_Y , i.e.:

$$\text{Is it possible that } X^* \geq_{\text{st}} Y \text{ even if } X_1 \leq_{\text{st}} Y_1 \text{ and } X_2 \leq_{\text{st}} Y_2? \quad (4)$$

In other words, recalling (2) and (3) we are looking for survival functions $\overline{F}_i(t)$ and $\overline{G}_i(t)$, $i = 1, 2$, such that

- (i) $[\overline{F}_1(t) + \overline{F}_2(t)]^2/4 \geq \overline{G}_1(t) \overline{G}_2(t)$ for all $t \geq 0$,
- (ii) $\overline{F}_i(t) \leq \overline{G}_i(t)$ for all $t \geq 0$, $i = 1, 2$.

Hereafter we characterize the solutions of the above problem in terms of a pair of functions that take values in the two-dimensional domain belonging to the first quadrant of \mathbb{R}^2 and limited by the orthogonal axis and by a time-varying branch of an equilateral hyperbola. Indeed, by setting

$$x_i(t) = \frac{\overline{G}_i(t)}{\overline{F}_i(t)} - 1, \quad t \geq 0, \quad i = 1, 2$$

relations (i) and (ii) are satisfied if and only if, for any fixed $t \geq 0$, we have

$$x_1(t) \geq 0, \quad x_2(t) \geq 0, \quad x_1(t) + x_2(t) + x_1(t) x_2(t) \leq A(t), \quad (5)$$

where we have set

$$A(t) := \frac{[\overline{F}_1(t) - \overline{F}_2(t)]^2}{4 \overline{F}_1(t) \overline{F}_2(t)}, \quad t \geq 0.$$

The answer to question (4) is affirmative. Indeed, two examples of non-empty families of solutions are given below.

Proposition 1 *Two sets of sufficient conditions such that inequalities (5) hold are the following:*

- (a) $x_1(t) = 0$ and $0 \leq x_2(t) \leq A(t)$, i.e.

$$\overline{G}_1(t) = \overline{F}_1(t) \quad \text{and} \quad \overline{F}_2(t) \leq \overline{G}_2(t) \leq \frac{[\overline{F}_1(t) + \overline{F}_2(t)]^2}{4 \overline{F}_1(t)}; \quad (6)$$

- (b) $x_1(t) = x_2(t)$ and $0 \leq x_2(t) \leq \sqrt{A(t) + 1} - 1$, i.e.

$$\frac{\overline{G}_1(t)}{\overline{F}_1(t)} = \frac{\overline{G}_2(t)}{\overline{F}_2(t)} \quad \text{and} \quad \overline{F}_1(t) \leq \overline{G}_1(t) \leq \frac{\overline{F}_1(t) + \overline{F}_2(t)}{2} \sqrt{\frac{\overline{F}_1(t)}{\overline{F}_2(t)}}. \quad (7)$$

Note that the functions appearing in the right-hand upper bounds of (6) and (7) are not necessarily survival functions.

Remark 1 Denoting the probability density functions of X_i and Y_i respectively by $f_i(x)$ and $g_i(x)$, $i = 1, 2$, we give two necessary conditions such that inequalities (5) hold:

$$f_1(0) = g_1(0) \quad \text{and} \quad f_2(0) = g_2(0). \quad (8)$$

Indeed, by setting $h(t) := [\overline{F}_1(t) + \overline{F}_2(t)]^2/4 - \overline{G}_1(t)\overline{G}_2(t)$ and $k_i(t) := \overline{G}_i(t) - \overline{F}_i(t)$, $i = 1, 2$, conditions (i) and (ii) imply the non-negativity of $h(t)$ and $k_i(t)$, $i = 1, 2$; moreover, since $h(0) = k_1(0) = k_2(0) = 0$, by imposing the non-negativity as $t \rightarrow 0^+$ of the derivatives of $h(t)$ and $k_i(t)$, $i = 1, 2$, we immediately obtain (8).

Example 1 A case in which conditions (6) are satisfied is the following:

$$\overline{F}_1(t) = \overline{G}_1(t) = e^{-\lambda t}, \quad \overline{F}_2(t) = (1 + \lambda t)e^{-\lambda t}, \quad \overline{G}_2(t) = \left[\left(1 + \frac{\lambda t}{2}\right)^2 - \left(\frac{\nu t}{2}\right)^2 \right] e^{-\lambda t},$$

for $t \geq 0$, where $\lambda > 0$ and $0 \leq \nu \leq \lambda$. In this case conditions (8) are satisfied, since $f_1(0) = g_1(0) = \lambda$ and $f_2(0) = g_2(0) = 0$. The mean values of the component lifetimes are given by

$$E(X_1) = E(Y_1) = \frac{1}{\lambda}, \quad E(X_2) = \frac{2}{\lambda}, \quad E(Y_2) = \frac{2}{\lambda} + \frac{1}{2\lambda} \left[1 - \left(\frac{\nu}{\lambda}\right)^2 \right],$$

so that $E(X_i) \leq E(Y_i)$, $i = 1, 2$. Moreover, we have that

$$E(X) = \frac{3}{4\lambda}, \quad E(Y) = \frac{13}{16\lambda} - \frac{1}{16\lambda} \left(\frac{\nu}{\lambda}\right)^2, \quad E(X^*) = \frac{13}{16\lambda}. \quad (9)$$

Hence, we obtain $E(X) \leq E(Y) \leq E(X^*)$ for all $0 \leq \nu \leq \lambda$, with $E(X) < E(Y) = E(X^*)$ when $\nu = 0$ and $E(X) = E(Y) < E(X^*)$ when $\nu = \lambda$.

Example 2 Let $\lambda > 0$, and let $u(t)$ be a non-negative function for $t \geq 0$, with $u(0) = 1$ and such that $\frac{d}{dt}u(t) \leq \lambda u(t)/2$, for $t \geq 0$. By taking survival functions that satisfy the following relations:

$$\overline{F}_1(t) = [u(t)]^2 e^{-\lambda t}, \quad \overline{F}_2(t) = e^{-\lambda t},$$

$$[u(t)]^2 e^{-\lambda t} \leq \overline{G}_1(t) \leq u(t) \frac{1 + [u(t)]^2}{2} e^{-\lambda t}, \quad \overline{G}_2(t) = \frac{\overline{G}_1(t)}{[u(t)]^2},$$

for $t \geq 0$, conditions (7) are fulfilled. For instance, by setting $u(t) = 1 + \lambda t/2$ and suitably choosing $\overline{G}_1(t)$ we have:

$$\overline{F}_1(t) = \left(1 + \frac{\lambda t}{2}\right)^2 e^{-\lambda t}, \quad \overline{F}_2(t) = e^{-\lambda t},$$

$$\overline{G}_1(t) = \left(1 + \frac{\lambda t}{2}\right) \left[1 + \frac{\lambda t}{2} + \frac{(\nu t)^2}{8}\right] e^{-\lambda t}, \quad \overline{G}_2(t) = \left[1 + \frac{(\nu t)^2/8}{1 + (\lambda t)/2}\right] e^{-\lambda t}, \quad (10)$$

for $t \geq 0$, where $\lambda > 0$ and $0 \leq \nu \leq \lambda$. Furthermore, from survival functions (10) it follows that $f_1(0) = g_1(0) = 0$ and $f_2(0) = g_2(0) = \lambda$, in agreement with (8).

Remark 2 The identity at the left-hand-side of (7) yields a relation between the hazard rate functions of the component random lifetimes. Indeed, denoting by

$$h_{\overline{F}}(t) = -\frac{d}{dt} \ln \overline{F}(t), \quad t \geq 0$$

the hazard rate function of a random lifetime characterized by the absolutely continuous survival function $\overline{F}(t)$, we have

$$\frac{\overline{G}_1(t)}{\overline{F}_1(t)} = \frac{\overline{G}_2(t)}{\overline{F}_2(t)} \quad \text{for all } t \geq 0$$

if and only if

$$h_{\overline{G}_1}(t) - h_{\overline{F}_1}(t) = h_{\overline{G}_2}(t) - h_{\overline{F}_2}(t) \quad \text{a.e.}$$

This relation is clearly satisfied by the hazard rates corresponding to survival functions (10).

We conclude this section by pointing out again that the affirmative answer to question (4) is not a paradox. Indeed, assumptions $X_1 \leq_{\text{st}} Y_1$ and $X_2 \leq_{\text{st}} Y_2$ imply $X^* \leq_{\text{st}} Y^*$, and this does not exclude the possibility that $X^* \geq_{\text{st}} Y$ in some cases.

3 A Parrondo's game

On the basis of the above results, and assuming more realistically that the independent random variables X_1, X_2, Y_1, Y_2 have finite mean values, we can build up a game in which a player gains the outcome of $X - Y$. Hence, he wins if a realization of X is larger than a realization of Y . Unfortunately, assumptions $X_1 \leq_{\text{st}} Y_1$ and $X_2 \leq_{\text{st}} Y_2$ lead to $X \leq_{\text{st}} Y$, and thus to a non-positive player's expected gain $E(X - Y) \leq 0$. We recall that in this case the allocation of units of system \mathcal{S}_X is fixed deterministically. However, let us assume that the game rules allow us to select randomly the units of \mathcal{S}_X , by means of mixtures with equal probabilities. This produces a new system lifetime X^* with survival function given in (3), so that the player now gains the outcome of $X^* - Y$. If the survival functions of X_1, X_2, Y_1, Y_2 satisfy conditions (i) and (ii) then $X^* \geq_{\text{st}} Y$, and the player's expected gain is non-negative, being $E(X^* - Y) \geq 0$. For instance, with reference to the case treated in Example 1, assuming $\lambda = 1$ and $\nu = 1/2$ from (9) we have

$$E(X) = \frac{3}{4}, \quad E(Y) = \frac{51}{64}, \quad E(X^*) = \frac{13}{16}.$$

Hence, the player's expected gain is

$$E(X - Y) = -\frac{3}{64} \quad \text{when the units of the first system are deterministically set}$$

and

$$E(X^* - Y) = \frac{1}{64} \quad \text{when the units of the first system are randomly mixed.}$$

Finally, this is a further confirmation of the standard conclusion of Parrondo's games: even if fixed settings lead to losses, we may expect a win in the presence of random mixing.

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